Additional Information for Chapter 3: Portfolio Choice and Mean-Variance Optimization

Advanced Microeconomics I

Andras Niedermayer[1]

[1]Department of Economics, University of Mannheim

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Utility Maximization Problem

- K assets
- $r \in \mathbb{R}^K$ asset returns (continuous or discrete distribution)
- expected value of r:

$$E[r] = \mu$$

- variance of r:

$$Var[r] := E[rr^T] - E[r]E[r^T] = \Sigma$$

Consumer's maximization problem:

$$\max_a E[u(\sum_k a_k r_k)]$$

$$\text{s.t.} \sum_k a_k = b$$

Quadratic Utility

Take quadratic utility function $u(x) = \beta x - x^2$, then

$$E[u(a \cdot r)] = \beta a \cdot \mu - a \cdot \mu\mu^T a - a \cdot \Sigma a$$

- only μ and Σ matter
- solution to utility maximization problem will have minimal variance $a \cdot \Sigma a$ for a given mean $a \cdot \mu$ (assuming $a \cdot 1 = 1$)

Mean-Variance Optimization

Unconstrained Optimization (a_k can be negative)

$$\min_a \quad \frac{1}{2} a \cdot \Sigma a$$
$$\text{s.t. } a \cdot 1 = 1$$
$$a \cdot \mu = \mu_p$$

Lagrangian

$$L = \frac{1}{2} a \cdot \Sigma a - \gamma (a \cdot 1 - 1) - \lambda (a \cdot \mu - \mu_p),$$

$\rightarrow$ Solve

$$\partial_a L = 0, \partial_\gamma L = 0, \partial_\lambda L = 0 \rightarrow a^*$$

Mean-Variance Optimization

Constrained Optimization (a_k *cannot* be negative)

$$\min_{a} \quad \frac{1}{2} a \cdot \Sigma a$$
$$\text{s.t. } a \cdot 1 = 1$$
$$a \cdot \mu = \mu_p$$
$$a \geq 0$$

Assume we know the set of assets with non-zero weights
$\mathbb{S} = \{k : a_k > 0\}$
Lagrangian

$$L = \frac{1}{2} a_S \cdot \Sigma_S a_S - \gamma(a_S \cdot 1_S - 1) - \lambda(a_S \cdot \mu_S - \mu_p),$$

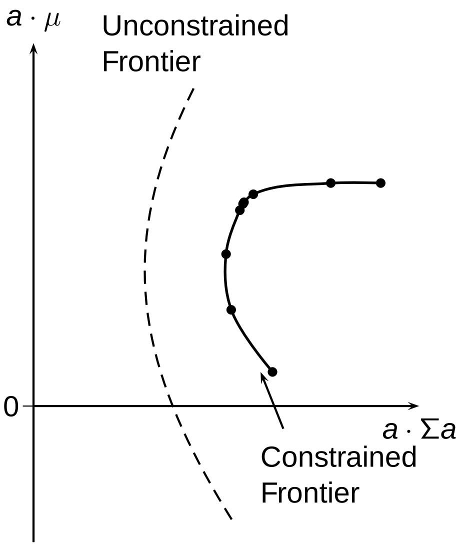
where $a_S > 0$

Using the constraint $a \cdot 1 = a_S \cdot 1_S = 1$ we have (for a given λ):

$$a_S^* = \lambda \Sigma_S^{-1} \left(\mu_S - \frac{1_S \cdot \Sigma_S^{-1} \mu_S}{1_S \cdot \Sigma_S^{-1} 1_S} 1_S \right) + \frac{\Sigma_S^{-1} 1_S}{1_S \cdot \Sigma_S^{-1} 1_S} \qquad (1)$$

- trying all 2^K possibilities to choose $\mathbb{S}$ would take too much time
- Niedermayer & Niedermayer (forthcoming) propose a fast computational method for solving this problem extending Markowitz's Critical Line Algorithm (CLA)

Constrained Efficient Frontier



Speed Comparison

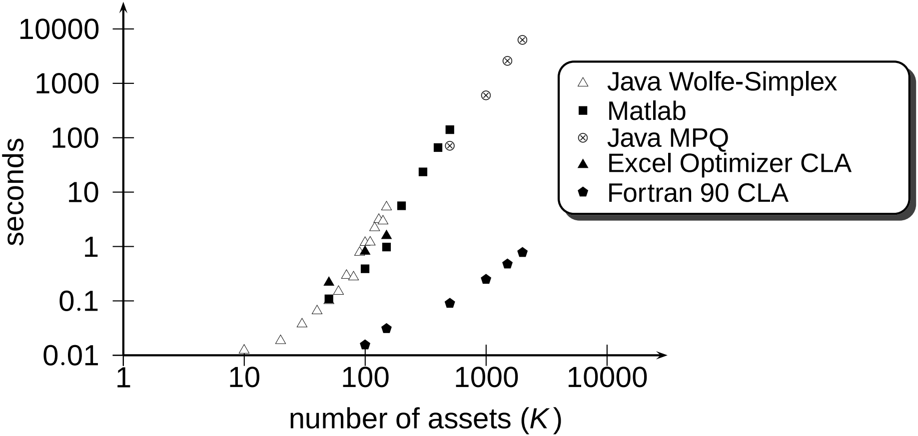


Figure: Testing different algorithms: CPU times for different number of assets and randomly generated positive definite covariance matrix.