Chapter 3: Individual Choice Under Uncertainty

Advanced Microeconomics I

Andras Niedermayer¹

¹Department of Economics, University of Mannheim

Fall 2009
so far: individual choices had completely predictable consequences

often choices where consumption involves uncertainty

Examples
- goods of uncertain quality
- savings decisions
- investment portfolios
- career moves
- insurance policies
- environmental policy choices

Question: how does a rational individual evaluate and compare risky choices?
Choices over baskets of goods under uncertainty are choices of probability distributions over $\mathbb{R}_+^H$.

Standard theory: different uncertain prospects and probability laws that they obey exogenously given to the individual decision maker (von Neumann and Morgenstern (1944))
**Definition:** A *lottery* $L$ is a probability distribution over $\mathbb{R}^H$ (or $\mathbb{R}_+^H$, if one cannot lose).

Important special cases:

- “Simple lotteries”: Lotteries over a fixed, finite subset of $\mathbb{R}_+^H$, $\{x_1, ..., x_S\}$:

  $$L = (\pi_1, ..., \pi_S) \in S^{S-1} = \{\pi \in \mathbb{R}_+^S | \sum_{s=1}^{S} \pi_s = 1\}.$$  

  ($S - 1$-dimensional simplex)

- “Money lotteries”: Lotteries over $\mathbb{R}$ (“money”): assume the existence of a given price system → reduce the goods space from $\mathbb{R}_+^H$ to $\mathbb{R}$.

- “Simple state lotteries”: for a fixed set of “states of nature” with probabilities $\pi_1, ..., \pi_S$ choose bundles in $\mathbb{R}_+^H$ for each state of nature: $L = (x_1, ..., x_S) \in \mathbb{R}_+^{HS}$. (See Section 3.2)
simple lotteries with $S = 3$ important for experiments and exposition $\rightarrow$ Marshak-Machina triangle

\[ \pi_2 = 1 - \pi_1 - \pi_3 = 0 \]
Example: Three possible monetary outcomes (all in Euro, say):
\[ x_1 = 0 \]
\[ x_2 = 500,000 \]
\[ x_3 = 2,500,000 \]

Here are two lotteries:
Lottery \( L^1 : (0, 1, 0) \)
\[ L^2 : (0.01, 0.89, 0.1) \]

Question: Which one do you prefer? (As an orientation: the expected values of these lotteries are 500,000 for the first and 695,000 for the second)
Here are two other lotteries (defined over the same outcomes $x_1 = 0; x_2 = 500,000; x_3 = 2,500,000$):

$L^3 : (0.89, 0.11, 0)$

$L^4 : (0.9, 0, 0.1)$

Question: Which one of these two do you prefer? (The expected values are 55,000 for $L^3$ and 250,000 for $L^4$) We return to the answer of this equation further below.
we look at choice under uncertainty using preference theory

\( \mathcal{L} \): the set of lotteries

\( \succsim \): a complete and transitive preference relationship on \( \mathcal{L} \)

typically more structure, e.g. for simple lotteries

\[ \mathcal{L} = S^{S-1} = \{ \pi \in \mathbb{R}^S_+; \sum_{s=1}^{S} \pi_s = 1 \} \]

the unit simplex

**Definition:** The preference relation \( \succsim \) is *continuous*, if for all \( L \in \mathcal{L} \) the sets \( \{ L' \in \mathcal{L}; L' \succsim L \} \) and \( \{ L' \in \mathcal{L}; L' \nsuccsim L \} \) are closed (in the relevant topology). By Proposition 2.1, \( \succsim \) can therefore be represented by a continuous utility function \( U : \mathcal{L} \rightarrow \mathbb{R} \). This means:

\[ L \succsim L' \in \mathcal{L} \iff U(L) \geq U(L') . \]
One can even use more structure of the space of all probability distributions, “linearity”:

**Definition:** Given $\alpha^1, \alpha^2 \geq 0$, $\alpha^1 + \alpha^2 = 1$, the mixture of lottery $L^1$ and $L^2$, $\alpha^1 L^1 + \alpha^2 L^2$, is the risky prospect which yields $L^i$ with probability $\alpha^i$, $i = 1, 2$.

- **Example simple lotteries:** If $L^i = (\pi^i_1, ..., \pi^i_S)$, then $\alpha^1 L^1 + \alpha^2 L^2 = (\alpha^1 \pi^1_1 + \alpha^2 \pi^2_1, ..., \alpha^1 \pi^1_S + \alpha^2 \pi^2_S)$.

- **Example money lotteries:** If $L^i$ is given by c.d.f. $F^i$, then $\alpha^1 L^1 + \alpha^2 L^2$ given by c.d.f. $\alpha^1 F^1 + \alpha^2 F^2$. 
Important (typical) restriction on preferences over $\mathcal{L}$:

**Definition:** A utility function $U : \mathcal{L} \rightarrow \mathbb{R}$ is called a *von Neumann-Morgenstern utility function*, if it is linear:

$$U(\alpha^1 L^1 + \alpha^2 L^2) = \alpha^1 U(L^1) + \alpha^2 U(L^2)$$

for all $\alpha^1, \alpha^2 \geq 0$, $\alpha^1 + \alpha^2 = 1$, and $L^i \in \mathcal{L}$. 
Proposition 3.1: (i) In the case of simple lotteries, utility functions are linear if and only if there are $S$ numbers $u_1, ... u_S$ such that for every $L = (\pi_1, ..., \pi_S) \in \mathcal{L}$

$$U(L) = \sum_{s=1}^{S} u_s \pi_s. \quad (1)$$

(ii) In the case of money lotteries, utility functions are linear if and only if there is a function $u : \mathbb{R} \to \mathbb{R}$ such that for every c.d.f. $F \in \mathcal{L}$

$$U(F) = \int_{\mathbb{R}} u(x) dF(x). \quad (2)$$

Intuition: If utility is linear, it can be patched together linearly from the “corner utilities” over certain events ($E_s$).
Proof.

for the finite case (i):  
1. (“if”, i.e. linear utility function is implied) Suppose that there are \( u_1, \ldots, u_s \) such that \( U \) can be represented as (1). Clearly, then \( U \) is linear.  
2. (“only if”, implied that there are \( u_1, \ldots, u_s \)) For any \( L \in \mathcal{L} \), let \( L = \sum_s \pi_s E_s \), where \( E_s = (0, \ldots, 1, \ldots, 0) \) is the degenerate lottery with unit weight at \( x_s \).  

...
Proof.

Then,

\[ U(L) = U \left( \sum_s \pi_s E_s \right) \]

\[ = U \left( \pi_1 E_1 + (1 - \pi_1) \sum_{s=2}^S \frac{\pi_s}{1 - \pi_1} E_s \right) \quad \text{(note: } \sum_{s=2}^S \frac{\pi_s}{1 - \pi_1} = 1) \]

\[ = \pi_1 U(E_1) + (1 - \pi_1) U \left( \sum_{s=2}^S \frac{\pi_s}{1 - \pi_1} E_s \right) \]

\[ = \ldots \]

\[ = \pi_1 U(E_1) + \ldots + \pi_S U(E_S) \]

and the conclusion follows by setting \( u_s = U(E_s) \) for all \( s \) (which does not depend on the specific \( L \)). Case (ii) homework.
Common assumptions:

- $u$ monotonically increasing
- $u$ continuous (for money lotteries)
Graphically, for linear preferences

\[ u_1 \pi_1 + u_2 \pi_2 + u_3 \pi_3 = \text{const} \quad \text{(substitute for } \pi_2 = 1 - \pi_1 - \pi_3)\]

\[ \iff (u_1 - u_2) \pi_1 + (u_3 - u_2) \pi_3 + u_2 = \text{const} \]

\[ \iff \pi_1 = \frac{\text{const} - u_2}{u_1 - u_2} + \frac{u_2 - u_3}{u_1 - u_2} \pi_3 \geq 0 \]
The previous figure shows that linearity of preferences (over lotteries) is quite a strong assumption, much more than what we have assumed for the deterministic case. In the latter, preferences \( U(x_1, x_2) = ax_1 + bx_2 \) would yield only extreme choices (see Figure 3.3).
Proposition 3.2: Suppose that the preference relation $\succeq$ over $\mathcal{L}$ is complete, transitive and continuous. Then it can be represented by a linear utility function iff

for all $L, L', L'' \in \mathcal{L}$ and $\alpha \in [0, 1]$:

$$L \succeq L' \iff \alpha L + (1 - \alpha)L'' \succeq \alpha L' + (1 - \alpha)L''$$  \hspace{1cm} (3)

Proof.

"$\Rightarrow$" trivial, "$\Leftarrow$" homework (not easy; easy for simple lotteries, see MWG and Gollier)

“Independence Axiom” (IA): if one mixes each of two lotteries $(L, L')$ with a third $(L'')$, then the ordering of the resulting mixtures is independent of $L''$
Intuition: Choice between \( \alpha L + (1 - \alpha)L'' \) and \( \alpha L' + (1 - \alpha)L'' \) is the same as

- flip a coin, probability \( 1 - \alpha \) tails → you get \( L'' \), \( \alpha \) head → either \( L \) or \( L' \)
- choose between \( L \) and \( L' \) before knowing head or tails
- if tails: choice didn’t matter
- if head: back to original choice between \( L \) and \( L' \)
Proposition 3.2: Suppose that $U : \mathcal{L} \rightarrow \mathbb{R}$ is a von Neumann-Morgenstern utility function for the preference relation $\succeq$ on the set of simple lotteries $\mathcal{L}$. Then $\tilde{U} : \mathcal{L} \rightarrow \mathbb{R}$ is another von Neumann-Morgenstern utility function for $\succeq$ if and only if there are scalars $\beta > 0$ and $\gamma$ such that $\tilde{U}(L) = \beta U(L) + \gamma$ for every $L \in \mathcal{L}$.

Proof.

Choose two lotteries $L$ and $\overline{L}$ with $L \succeq L \succeq \overline{L}$ for all $L \in \mathcal{L}$. If $L \sim \overline{L}$ every utility function must be constant. The result then follows immediately. Next we look at $\overline{L} \succ L$. ...

Proof.

Write $L$ as $L = \sum_s \pi_s E_s$, where $E_s = (0, \ldots, 1, \ldots, 0)$ is the degenerate lottery with unit weight at $x_s$. $U$ is a von Neumann-Morgenstern utility function and $\tilde{U}(L) = \beta U(L) + \gamma$. To be shown: $\tilde{U}$ is also a von Neumann-Morgenstern utility function:

...
Proof.

\[ \tilde{U}(L) = \tilde{U}\left(\sum_s \pi_s E_s\right) = \beta U\left(\sum_s \pi_s E_s\right) + \gamma \]

\[ = \beta \left[ \sum_s \pi_s U(E_s) \right] + \gamma \]

\[ = \sum_s \pi_s (\beta U(E_s)) + \sum_s \pi_s \gamma \]

\[ = \sum_s \pi_s (\beta U(E_s) + \gamma) \]

\[ = \sum_s \pi_s \tilde{U}(E_s) \]

Hence \( \tilde{U} \) has a von-Neumann-Morgenstern utility representation. Further, \( \tilde{U} \) represents the same preferences as \( U \).
Proof.

Opposite direction of the proof: $U$ and $\tilde{U}$ are von Neumann-Morgenstern utility functions representing $\succeq \Rightarrow \tilde{U}(L) = \beta U(L) + \gamma$ for all $L \in \mathcal{L}$. (to be shown)
Take any lottery $L \in \mathcal{L}$ and define $\alpha_L$ by

$$U(L) = \alpha_L U(\bar{L}) + (1 - \alpha_L) U(L),$$

$\iff$

$$\alpha_L = \frac{U(L) - U(\bar{L})}{U(L) - U(\underline{L})}.$$

$\alpha_L U(\bar{L}) + (1 - \alpha_L) U(L) = U(\alpha_L \bar{L} + (1 - \alpha_L) \underline{L})$ implies $L \sim \alpha_L \bar{L} + (1 - \alpha_L) \underline{L}$. 
Proof.

\( \tilde{U} \) represents same preferences as \( U \) \( \Rightarrow \)

\[
\begin{align*}
\tilde{U}(L) &= \tilde{U}(\alpha_L \bar{L} + (1 - \alpha_L)\underline{L}) \\
&= \alpha_L \tilde{U}(\bar{L}) + (1 - \alpha_L)\tilde{U}(\underline{L}) \\
&= \alpha_L [\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})] + \tilde{U}(\underline{L}).
\end{align*}
\]

Substituting for \( \alpha_L \) we obtain \( \tilde{U}(L) = \beta U(L) + \gamma \) where

\[
\begin{align*}
\beta &= \frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(L) - U(\underline{L})}, \text{ and} \\
\gamma &= \tilde{U}(\underline{L}) - U(L) \frac{\tilde{U}(\bar{L}) - \tilde{U}(\underline{L})}{U(L) - U(\underline{L})}
\end{align*}
\]
Independence Axiom (⇔ linear utility) controversial, because of experimental evidence, e.g. Battaglio, Kagel, Jiranyakul (J. Risk Uncertainty, 1990):

Three pairs of simple lotteries on \( \{0, 12, 20\} \)

\[
\begin{align*}
S^1 &= (0, 0.4, 0.6), & R^1 &= (0.16, 0, 0.84) \\
S^2 &= (0, 1, 0), & R^2 &= (0.4, 0, 0.6) \\
S^3 &= (0.8, 0.2, 0), & R^3 &= (0.88, 0, 0.12).
\end{align*}
\]

Graphically, in Marschak-Machina triangle:
indifference curves are straight lines \implies\text{ either } S^i \succeq R^i \text{ for all } i \text{ or } R^i \succeq S^i \text{ for all } i

However, in experiments non-homogeneous choices (not only \( S \), not only \( R \))
Example: Allais Paradox: Take the lotteries we have looked at before: $L^1 = (0, 1, 0), L^2 = (0.01, 0.89, 0.1), \text{ and } L^3 = (0.89, 0.11, 0), L^4 = (0.9, 0, 0.1)$ on payments $x_1 = 0; x_2 = 500,000; x_3 = 2,500,000$, graphically:
With expected utility, graphically:

- preferences are either relatively steep (see figure), $\rightarrow L^1$ and $L^3$ will be preferred.
- Or: preferences are relatively flat $\rightarrow L^2$ and $L^4$ are preferred (of course, individuals may be indifferent).
Mathematically:
Take $L^A = (0, 1, 0)$, $L^B = \left(\frac{1}{11}, 0, \frac{10}{11}\right)$

<table>
<thead>
<tr>
<th>$L^1$</th>
<th>$L^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.89(0, 1, 0) + 0.11L^A$</td>
<td>$0.89(0, 1, 0) + 0.11L^B$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L^3$</th>
<th>$L^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.89(1, 0, 0) + 0.11L^A$</td>
<td>$0.89(1, 0, 0) + 0.11L^B$</td>
</tr>
</tbody>
</table>

In experiments: large part (often majority) $L^1 \succeq L^2$ and $L^4 \succeq L^3 \rightarrow$ indifference curves “fan out”
In most applications, one still works with linear utility, because

- it is conceptually simple,
- people have always done it,
- it allows simple derivations of concepts such as risk aversion, etc.
- choice under uncertainty is indeed different from choice under certainty: the linear combination of lotteries considered in the Independence Axiom is not actually consumed (at least when you view it as a two-stage lottery): the comparison is rather between preferring $L$ to $L'$ ex-ante or ex-post. In the context of choice under certainty, linear combinations of bundles are consumed as combinations.
theory of “state lotteries” or of lotteries as contingent plans (similar to theory of choice under certainty as discussed in Chapter 2)

Here: for the finite case (“simple state lotteries”)

- fix a set of probabilities $\pi_1, \ldots, \pi_S \geq 0, \sum_{S} \pi_s = 1$,
- vary the consumption bundles $x_s \in \mathbb{R}_+^H$ that the individual can get with these probabilities.
- Formally, the set of all these lotteries is identical to $(\mathbb{R}_+^H)^S$, the set of all $x = (x_1, \ldots, x_S)$ with $x_s \in \mathbb{R}_+^H$.

Interpretation: $S$ “states of nature”, with probabilities $\pi_s$. Choice today: “contingent consumption bundle” (a plan for state-dependent consumption).
Lotteries and Expected Utility

Lotteries as Contingent Plans

- complete, transitive, and continuous preferences $\succsim$ over these plans $\Rightarrow$ continuous utility function $U$ representing $\succsim$

- Note: $\mathcal{L} = \mathbb{R}^{HS}$ is a natural linear space, (for any $x, y \in \mathcal{L}$ and $\alpha, \beta \in \mathbb{R}_+ \Rightarrow \alpha x + \beta y \in \mathcal{L}$)

- interpretation is natural (the lottery $\alpha x + \beta y$ yields $\alpha x_s + \beta y_s$ in each state), but different from the general definition of linear combinations given earlier

- earlier definition: different form of linear combination: mixing two state lotteries $x$ and $y$ with random binary variable (probabilities $\alpha$ and $1 - \alpha$) $\rightarrow$ binary lottery involving $x_s$ and $y_s$ in each state $s$
Using this notion (linearity in probabilities instead of linearity in quantities), allows us to retrace the approach of simple lotteries taken earlier. Independence Axiom:

**Definition:** A utility function over contingent plans satisfies the Independence Axiom if, for any \( x, x' \in \mathbb{R}^{HS}_+, y_s \in \mathbb{R}^H_+, z_s \in \mathbb{R}^H_+ \),

\[
U(x_s y_s) \geq U(x'_s y_s) \text{ if and only if } U(x_s z_s) \geq U(x'_s z_s)
\]

Here, \( x_s y_s \) is the contingent plan \( x \) with \( x_s \) replaced by \( y_s \).

Analog to Prop. 3.2

**Proposition 3.4:** (Additivity) Assume that there are at least three states. If and only if \( \succsim \) satisfies the Independence Axiom, there exist state-dependent “elementary utility functions” \( u_s : \mathbb{R}^H_+ \rightarrow \mathbb{R} \) such that

\[
U(x) = \sum_s u_s(x_s) \text{ for all } x.
\]
So far we allowed the preferences of an individual to depend on state $s$. Disallowing this we get the full analog of
\[ U(L) = \sum_{s=1}^{S} u_s \pi_s \] shown previously:

**Proposition 3.5:** (Linearity) If, in addition, $U$ does not depend on $s$ directly, but only on the probability distribution of the $x_s$ (i.e. on the $\pi_s$), then there is a $u : \mathbb{R}_+^H \rightarrow \mathbb{R}$ such that

\[ U(x) = \sum_s \pi_s u(x_s) \] for all $x$. 
• state independence of preferences is a strong assumption (e.g. insurance/accidents)
• but: with larger space of consumption bundle → preferences depending on state directly
• alternatively: parameterize $u_s$ by simple parameter ("endowment loss") (especially for monetary model $H = 1$)
relationship between two concepts "simple lottery" and "state lottery":

- **state lottery**: choose a lottery \((x_1, \ldots, x_S; \pi_1, \ldots, \pi_S)\), choice is among \((x_1, \ldots, x_S) \in \mathbb{R}^{HS}^+,\) the \(\pi_S\) are fixed

- **simple lottery model**: choose a lottery \([x_1, \ldots, x_S; \pi_1, \ldots, \pi_S]\), choice is among the \((\pi_1, \ldots, \pi_S) \in S^{S-1}\), the \(x_S\) are fixed

**Focus**

- empirical literature: mostly simple lottery, replacing Expected Utility hypothesis

- theoretical General Equilibrium literature: state-lottery model
Application 1: Demand for Insurance

- $H = 1$ (monetary world)
- $S = 2$, where $s = 1 \leftrightarrow$ accident, $s = 2 \leftrightarrow$ no accident
- state-contingent endowments $e_1 < e_2$
- before endowments materialize: market in state-contingent claims (monetary payments)
- $p$: price of state-1 consumption in terms of state-2 consumption (i.e. $p = p_1/p_2$: “how many units of state-2 consumption must be given for 1 unit of state-1 consumption”)
- utility function

$$U(x_1, x_2) = u_1(x_1) + u_2(x_2)$$
optimal allocation of endowments:

\[
\max_{x_1, x_2} u_1(x_1) + u_2(x_2)
\]
\[
\text{s.t. } px_1 + x_2 \leq pe_1 + e_2
\]

- with monotonicity "\(\leq\)" → "\(=\)"
- with differentiability and \(u'_i(0) = \infty\): → \(u'_1(x_1) = pu'_2(x_2)\)
  (marginal rate of substitution = price):

\[
\frac{u'_1(x_1)}{u'_2(x_2)} = p
\]

→optimal choice of insurance (however, you may have \(x_2 > e_2, x_1 < e_1\))
Additional restriction: linearity. This yields

\[ u_i(x_i) = \pi_i u(x_i), \ i = 1, 2 \]

\[ \implies \frac{\pi_1 u'(x_1)}{\pi_2 u'(x_2)} = p \]

\( \pi_i \): subjective/objective probabilities for state \( i \).

- \( p = \frac{\pi_1}{\pi_2} \) (“actuarially fair insurance”) \( \Rightarrow x_1 = x_2 \) (if \( u' \) monotone)
- \( p \neq \frac{\pi_1}{\pi_2} \) \( \Rightarrow \) demand depends on shape of \( u \)
- if \( u \) concave

\[ U(x_1, x_2) = \pi_1 u(x_1) + \pi_2 u(x_2) < u(\pi_1 x_1 + \pi_2 x_2) = U(\pi_1 x_1 + \pi_2 x_2, \pi_1 x_1 + \pi_2 x_2) \]

- decision maker equalizes over states
- for \( p > \frac{\pi_1}{\pi_2} \) (“actuarially unfair insurance”) we have \( x_1 < x_2 \) (under-insurance)
Portfolio Choice

Assumptions

- Monetary model: $H = 1$
- $S$ states of nature, probabilities $\pi_s$.
- $U$ does not depend on state directly.
- $K$ assets (securities) that can be purchased before the state of nature is realized. Asset $k$ pays off $v^k_s \in \mathbb{R}$ in state $s$.
- Asset prices $q \in \mathbb{R}^K$
- budget $b$ to spend on assets, no endowments in the different states
Individual’s problem: choose asset holdings $z \in \mathbb{R}^K$ to maximize expected utility subject to budget constraint:

$$\max_{x,z} U(x) = \sum_s \pi_s u(x_s)$$

s.t. $x_s = \sum_k z^k v^k_s$

$$\sum_k q^k z^k = b$$

- negative consumption or negative asset holdings ("short sales") allowed
Matrix notation: Let

$$V = (v^1, ..., v^K) = \begin{pmatrix} v_1^1 & \cdots & v_1^K \\ \vdots & \ddots & \vdots \\ v_S^1 & \cdots & v_S^K \end{pmatrix}$$

be the payoff matrix of the $K$ assets. Then the problem is

$$\max_x \sum_s \pi_s u(x_s)$$

subject to

$$x = Vz$$

and

$$q \cdot z = b$$

(4)
Special cases:

- Arrow securities: There are \( K = S \) assets, and asset \( k \) pays 1 unit in state \( k \) and 0 in all other states:

\[
V = \begin{pmatrix}
1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 1 \\
\end{pmatrix}
\]

- Redundant securities: A security \( k \) is redundant if there are securities \( k_1, \ldots, k_m \neq k \) and numbers \( \alpha_i \) such that

\[
V^k = \sum_{i=1}^{m} \alpha_i V^{k_i}.
\]
Complete asset market: The set of securities is complete (constitutes a complete asset market) if the payoff matrix $V$ has rank $S$: for each $x \in \mathbb{R}^S$ there is a $z$ such that $x = Vz$. Eliminate redundant securities (not unique) → $V$ is full rank $S \times S$ matrix, and the utility maximization problem can be written as

$$\max_x \sum_s \pi_s u(x_s)$$

s.t. $q \cdot V^{-1}x = b$

The risk-free asset: An asset that has the same payoff in each state: $v^k_s = \bar{v}$. (Note: why "the" risk-free asset?)
**Remark:** complete asset market $\Rightarrow$ there is always a risk-free asset (portfolio with constant payoff):

$$z = V^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Often: label this portfolio “asset 1", and ignore one of the original assets.
(This can also work for incomplete asset market)
A reformulation of the portfolio problem: Let $a^k = q^k z^k$ denote the amount of wealth invested in asset $k$. Let $r^k = v^k / q^k$ be the (gross) return of asset $k$. Then problem (4) can be written as

$$\max_a Eu\left(\sum_{k=1}^{K} a^k r^k\right)$$

s.t. $\sum_{k=1}^{K} a^k = b$
here: $H = 1$ (monetary allocations).

Interpretation: $U$ over lotteries is an indirect utility function for a given set of prices.

⇒ von Neumann-Morgenstern utility is defined over “income” $x \in \mathbb{R}$

set of all possible "states": $\mathbb{R} \Rightarrow \text{“state model” and the “lottery model” are identical:}

  - in the “lottery model” one chooses a cumulative distribution function (c.d.f.) $F$,
  - in the “state model” one has a c.d.f. $G$ over $\mathbb{R}$ (the states) and choose $h : \mathbb{R} \to \mathbb{R}$ (states $\to$ incomes), which yields a c.d.f. over $\mathbb{R}$ (income), as well.
Following the discussion of the last section, we will consider a decision maker who has preferences over the space

$$\mathcal{L} = \{ F : \mathbb{R} \to [0, 1] ; F(-\infty) = 0, F(\infty) = 1, \ F \text{ continuous from the right} \}$$

and whose preferences can be represented by a linear utility function $U$:

$$U(F) = \int_{\mathbb{R}} u(x) dF(x).$$

$u$ is continuous and increasing ($\Leftrightarrow$ monotonicity in the certain case). $\rightarrow$ “money-utility function”
Digression: The St.-Petersburg-Paradox:

Suppose an individual has vNM utility with unbounded money-utility. Let \( x_n \) be defined by \( u(x_n) = 2^n, n \geq 1 \).

(e.g. \( u(x) = \sqrt{x} \) \( \implies x_n = 4^n, \) and \( x_{10} = 1,048,576 \))

Following lottery: Coin tossed repeatedly, until “heads” comes up. If it comes up at the \( n \)-th toss, the individual gets paid \( x_n \). Individuals willingness to pay for this lottery:

\[
U(F) = \sum_{n=1}^{\infty} \frac{1}{2^n} u(x_n) = \sum_{n=1}^{\infty} 1 = \infty
\]

Question: Is this reasonable?
Answer: Probably not. This phenomenon does not occur if \( u \) is bounded above. (But note that in applications these types of unbounded gambles never occur)
**Definition:** For an individual with money-utility $u$, the certainty equivalent of a monetary lottery $F$, $c(F, u)$, is defined as

$$u(c(F, u)) = \int_{\mathbb{R}} u(x) dF(x)$$

$c(F, u)$ is the amount of money which the individual, if obtained for sure, values as equivalent to the lottery $F$. 
Definition:
An individual is risk averse if \( c(F, u) < \int x\,dF(x) \) for all \( F \),
An individual is risk neutral if \( c(F, u) = \int x\,dF(x) \) for all \( F \),
An individual is risk loving if \( c(F, u) > \int x\,dF(x) \) for all \( F \).
→“global” risk attitude: for all wealth levels and all lotteries. Alternatively: an individual attitude towards small risks around a given wealth level:

**Definition:** Let \( \varepsilon \) be a random variable with mean 0 and c.d.f. \( \phi \). The risk premium (demanded by a given individual) for gamble \( \varepsilon \) at wealth level \( x \), \( \rho(x, \varepsilon) \), is given by

\[
u(x - \rho(x, \varepsilon)) = \int_R u(x + \varepsilon) d\phi(\varepsilon).
\]

In other words: \( x - \rho(x, \varepsilon) \) is the certainty equivalence of \( x + \varepsilon \).
**Proposition 3.6:** For an individual with vNM utility the following statements are equivalent:

(i) The individual is risk averse.
(ii) $\rho(x, \varepsilon) > 0$ for all $x, \varepsilon$ with $E\varepsilon = 0$.
(iii) $u$ is strictly concave.
Proof.

(i) $\implies$ (ii): Take any $x, \tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$. Since $x - \rho(x, \tilde{\varepsilon})$ is the certainty equivalent of $x + \tilde{\varepsilon}$, (i) implies

$$x - \rho(x, \tilde{\varepsilon}) < \int (x + \varepsilon)d\phi(\varepsilon)$$

$$= x + E\tilde{\varepsilon}$$

$$= x$$

$$\implies \rho(x, \tilde{\varepsilon}) > 0$$
(ii) $\implies$ (iii): Take any $y, z \in \mathbb{R}$ and $\alpha \in [0, 1]$. Let $x = \alpha y + (1 - \alpha)z$. Define the gamble

$\tilde{\varepsilon} = \begin{cases} y - x & \text{with proba } \alpha \\ z - x & \text{with proba } 1 - \alpha \end{cases}$

Then

$$\int u(x + \varepsilon) d\phi(\varepsilon) = \alpha u(y) + (1 - \alpha)u(z).$$

...
Proof.

By the definition of $\rho$,

$$u(x - \rho(x, \tilde{\varepsilon})) = \int u(x + \varepsilon) d\phi(\varepsilon)$$

Hence, $u(x - \rho(x, \tilde{\varepsilon})) = \alpha u(y) + (1 - \alpha) u(z)$. Since $u$ is strictly increasing, (ii) implies

$$u(x) > \alpha u(y) + (1 - \alpha) u(z).$$
Proof.

(iii)→(i): Remember Jensen’s inequality: If $u$ is strictly concave,

$$u \left( \int x dF(x) \right) > \int u(x) dF(x)$$

for all non-trivial $F$.

Suppose that $\int x dF(x) \leq c(F, u)$ for one $F$ (i.e. no risk aversion). Since $u$ is increasing and strictly concave

$$\int u(x) dF(x) \underset{\text{Jensen}}{<} u \left( \int x dF(x) \right) \underset{\text{u increasing}}{\leq} u(c(F, u))$$

which is a contradiction to the definition of $c$.\qed
Measuring Risk Aversion

Two main questions:

1. Can one compare the risk aversion of different individuals? In particular, can one give a sense to the statement “i is more risk averse than j”?

2. Can one define a “measure of risk aversion”, which would not only allow to compare different utility functions, but also to give sense to statements such as “i has very high risk aversion”, “j has risk aversion of around 4”? 
Question 1: Can one compare the risk aversion of different individuals? In particular, can one give a sense to the statement “i is more risk averse than j”?

Two obvious candidates for answers are

- “\( u_i \) is more concave than \( u_j \)”
- “at all wealth levels and for all gambles \( \tilde{\epsilon} \), i demands a higher risk premium”

**Proposition 3.7:** Consider two money-utility functions \( u_1 \) and \( u_2 \). The following two statements are equivalent:
(i) There exists a strictly concave function \( \psi \) such that \( u_1(x) = \psi(u_2(x)) \) for all \( x \)
(ii) \( \rho_1(x, \tilde{\epsilon}) > \rho_2(x, \tilde{\epsilon}) \) for all \( x \) and \( \tilde{\epsilon} \) with \( E\tilde{\epsilon} = 0 \).
Proof.

(i)⇒(ii): By the definition of the risk premium, 

\[ u_i(x - \rho_i(x, \tilde{\varepsilon})) = \int u_i(x + \varepsilon) d\phi(\varepsilon). \]

Since \( u_i \) is strictly increasing and thus can be inverted, we have

\[ \rho_i(x, \tilde{\varepsilon}) = x - u_i^{-1}\left(\int u_i(x + \varepsilon) d\phi(\varepsilon)\right) \]
Proof.

Hence,

\[
\rho_1(x, \varepsilon) - \rho_2(x, \varepsilon) = u_2^{-1} \left( \int u_2(x + \varepsilon) d\phi(\varepsilon) \right) - u_1^{-1} \left( \int u_1(x + \varepsilon) d\phi(\varepsilon) \right) 
\]

\[
= u_2^{-1} \left( \int u_2(x + \varepsilon) d\phi(\varepsilon) \right) - u_1^{-1} \left( \int \psi(u_2(x + \varepsilon)) d\phi(\varepsilon) \right) 
\]
Proof.

By Jensen’s inequality (applied to the strictly concave function $\psi$, after change of variable $\eta = u_2(x + \varepsilon)$, note: $u'_2 > 0$):

$$\int \psi(u_2(x + \varepsilon))d\phi(\varepsilon) < \psi(\int u_2(x + \varepsilon)d\phi(\varepsilon))$$

Hence, because $u_1^{-1}$ is increasing,

$$\rho_1(x, \bar{\varepsilon}) - \rho_2(x, \bar{\varepsilon}) > u_2^{-1}\left(\int u_2(x + \varepsilon)d\phi(\varepsilon)\right) - u_1^{-1}\left(\psi(\int u_2(x + \varepsilon)d\phi(\varepsilon))\right)$$
Proof.

Since

\[ u_1(x) = \psi(u_2(x)) \iff u_2^{-1}(u) = u_1^{-1}\psi(u) \]

the right hand side of the last inequality is 0. This completes the first part of the proof.

(ii) \(\implies\) (i): homework.
Question 2 (more general): Can one define a “measure of risk aversion”, which would not only allow to compare different utility functions, but also to give sense to statements such as “i has very high risk aversion”, “j has risk aversion of around 4”?

natural question: "what is the risk premium per unit of variance of a gamble that a risk averse individual demands?"
consider a “small gamble” (all mass close to 0) $\tilde{\varepsilon}$.

How much is the individual ready to pay in order to avoid having the random wealth $x + \tilde{\varepsilon}$?

By Taylor expansion

$$u(x + \varepsilon) \approx u(x) + \varepsilon u'(x) + \frac{\varepsilon^2}{2} u''(x) \quad \text{for small } \varepsilon \in \mathbb{R} \quad (6)$$

$$\Rightarrow \int u(x + \varepsilon) \, d\phi(\varepsilon) \approx u(x) + \frac{1}{2} u''(x) \text{var}(\tilde{\varepsilon})$$
On the other hand, also by Taylor expansion (since for $\tilde{\varepsilon}$ small, $\rho(x, \tilde{\varepsilon})$ is small),

$$u(x - \rho(x, \tilde{\varepsilon})) \approx u(x) - \rho(x, \tilde{\varepsilon})u'(x)$$

By the definition of $\rho$:

$$u(x) - \rho(x, \tilde{\varepsilon})u'(x) \approx u(x) + \frac{1}{2}u''(x)\text{var}(\tilde{\varepsilon})$$

$$\iff \frac{\rho(x, \tilde{\varepsilon})}{\text{var}(\tilde{\varepsilon})} \approx \frac{1}{2} \frac{u''(x)}{u'(x)}$$

Hence, for “small gambles around $x$”, the risk premium per unit of variance demanded by a risk averse individual is approximately $-\frac{1}{2} \frac{u''(x)}{u'(x)}$. 
Definition: If the money-utility function of a utility maximising individual is twice differentiable,

\[ r^A(x) = -\frac{u''(x)}{u'(x)} \]

is called the coefficient of absolute risk aversion at \( x \).
Proposition 3.8: If the two money-utility functions $u_1$ and $u_2$ are twice differentiable, then the following two statements are equivalent:

(i) There exists a strictly concave function $\psi$ such that $u_1(x) = \psi(u_2(x))$ for all $x$

(ii) $r_A^1(x) > r_A^2(x)$ for all $x$. 
Proof.

We can always write \( u_1(x) = \psi(u_2(x)) \).
Since \( u_1 \) and \( u_2 \) strictly increasing, \( \psi \) must be so as well.
Differentiating:

\[
\begin{align*}
u_1'(x) &= \psi'(u_2(x))u_2'(x) \\
u_1''(x) &= \psi''(u_2(x))(u_2'(x))^2 + \psi'(u_2(x))u_2''(x)
\end{align*}
\]

Divide the second by the first equation:

\[
r_1^A(x) = -\frac{\psi''(u_2(x))}{\psi'(u_2(x))}u_2'(x) - \frac{u_2''(x)}{u_2'(x)}
\]

\[
= -\frac{\psi''(u_2(x))}{\psi'(u_2(x))}u_2'(x) + r_2^A(x)
\]

Hence, \( r_1^A > r_2^A \iff \psi'' < 0 \).
The most important comparative statics property of $r^A$ concerns the question how the attitude towards risk is influenced by wealth:

**Proposition 3.9:** An individual’s money-utility function exhibits decreasing absolute risk aversion (i.e., $r^A \downarrow$ in $x$) if and only if her risk premium is decreasing in wealth ($\rho(x, \tilde{\varepsilon}) \downarrow$ in $x$) for every gamble $\tilde{\varepsilon}$.

**Proof:** Exercise.
instead of additive gamble $x + \tilde{\varepsilon}$ in (6) → multiplicative gamble $(1 + \tilde{\varepsilon})x$ (similar approximation as above):

maximum *share of her wealth* that a risk averse individual is ready to pay in order not to lose or gain a random *share* $\tilde{\varepsilon}$ of her wealth is approximately

$$-\frac{1}{2} \text{var}(\tilde{\varepsilon}) \frac{xu''(x)}{u'(x)}.$$ 

→ relative risk premium.
**Definition:** If the money-utility function of a utility maximizing individual is twice differentiable,

\[ r^R(x) = -\frac{xu''(x)}{u'(x)} \]

is called the *coefficient of relative risk aversion* at \( x \).

**Note:**

- coefficient of relative risk aversion: no dimension
- coefficient of absolute risk aversion: dimension of 1/money unit
For computational reasons, the following utility functions are often used:

**CARA** \( u(x) = -e^{-ax}, a > 0 \). This function has constant absolute risk aversion with coefficient \( a \).

**CRRA**
\[
u(x) = \begin{cases} 
  \frac{1}{1-\gamma} x^{1-\gamma} & \text{if } \gamma \neq 1, \gamma > 0 \\
  \ln x & \text{if } \gamma = 1
\end{cases}
\]
This function has constant relative risk aversion with coefficient \( \gamma \).

**LRT** \( u(x) = \frac{1}{\gamma-1} (\alpha + \gamma x)^{1-1/\gamma}, \text{ for } \alpha + \gamma x > 0 \). This function has linear (better: affine) risk tolerance (which means that the inverse of its absolute risk aversion is linear in wealth). They are also called HARA utility functions (hyperbolic absolute risk aversion - because absolute risk aversion as a function of wealth describes a hyperbola).
**Proposition 3.10:** Consider the portfolio choice problem of Section 3.2, assume that there are $K$ assets, and that asset 1 is risk-free with return $\bar{r}$. If an agent has utility with linear risk tolerance (LRT), then the solution $a$ of his portfolio choice problem (5) is linear in wealth. This means: there exist numbers $\lambda^k, \mu^k \in \mathbb{R}$ such that

$$a^k = \lambda^k + \mu^k b, \quad k = 1, ..., K.$$ 

**Proof:** Exercise.
introduced to economics by Rothschild and Stiglitz (JET 1970) Denote $F(x)$ and $G(x)$ two c.d.f. ($F$ and $G$ have to be upper-semicontinuous); both are money lotteries.

**Definition:** The distribution function $F$ *first-order stochastically dominates* distribution function $G$ if

$$
\int_0^\infty u(x) dF(x) \geq \int_0^\infty u(x) dG(x) \tag{7}
$$

for every nondecreasing function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$. The distribution function $F$ *second-order stochastically dominates* distribution function $G$ if $E_F x = E_G x$ and (7) holds for every concave function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$. 
Interpretation

- first-order stochastic dominance: no expected utility maximizer will prefer \( G \) to \( F \).
- second-order stochastic dominance: consider two lotteries with the same mean. No risk-averse expected utility maximizer prefers \( G \) to \( F \).
Because of the following proposition stochastic dominance can be defined without having to explicitly specify utilities. 

**Proposition 3.11**: (i) The distribution function $F$ first-order stochastically dominates distribution function $G$ if and only if, for all $x \geq 0$, $F(x) \leq G(x)$.

(ii) The distribution function $F$ second-order stochastically dominates distribution function $G$ if and only if they have the same mean and the Lorenz curve of $G$ does not lie below the Lorenz curve of $F$,

$$
\int_0^x F(y)dy \leq \int_0^x G(y)dy
$$

for all $x \geq 0$.

The proof is a bit lengthy.