Definition: A preference relation on the set $X$ is a binary relationship, denoted $\succeq$, where $x \succeq y$ reads “$x$ is at least as good as $y$”.
Standard assumptions on preferences in economics:

- **Completeness**: For all \( x, y \in X \), either \( x \succeq y \) or \( y \succeq x \), or both.

- **Transitivity**: For all \( x, y, z \in X \), if \( x \succeq y \) and \( y \succeq z \), then \( x \succeq z \).

- **Local non-satiation**: Suppose the \( X \) has a metric. For all \( x \in X \) and all \( \varepsilon > 0 \), there is a \( y \in X \) such that \( \| y - x \| < \varepsilon \) and \( y \succ x \).

- **Convexity**: For all \( x \in X \), the upper contour set \( \{ x' \in X; x' \succeq x \} \) is convex.

- **Continuity**: For all \( x \in X \), the contour sets \( \{ x' \in X; x' \succeq x \} \) and \( \{ x' \in X; x' \preceq x \} \) are closed sets.
Two special classes of preferences:

- Homotheticity: A preference relation is homothetic if $x \sim y$ implies $\lambda x \sim \lambda y$ for all $\lambda > 0$.

- Quasi-linearity: A preference relation on $X \subset \mathbb{R}^H$ is quasi-linear with respect to commodity 1 if $x \sim y$ implies $x + \lambda e_1 \sim y + \lambda e_1$ for all $x, y, x + \lambda e_1, y + \lambda e_1 \in X$ and $\lambda \in \mathbb{R}$.
Proposition 2.1

If $\succeq$ is complete, transitive, and continuous, then $\succeq$ can be represented by a continuous utility function, i.e., there exists a continuous function $u : X \to \mathbb{R}$ such that for any $x, y \in X$, $x \succeq y \iff u(x) \geq u(y)$. 
2.3 Utility Maximization

Given a budget $b$ and price $p \gg 0$, the consumer solves

$$\max_x u(x)$$

s.t. $x \in X$ \hspace{1cm} (UM) \hspace{1cm} (1)

$$p \cdot x \leq b$$

**Definition:** The set of all solutions of (UM) for a given $(p, b)$ is denoted $\varphi(p, b)$, and $\varphi$ is called the Marshallian demand correspondence. For $x \in \varphi(p, b)$, $v(p, b) = u(x)$ is called the indirect utility function.
Proposition 2.2

If $\succeq$ is locally non-satiated and $u$ continuous on $X = \mathbb{R}^H_+$, then

$(H\varphi)$ $\varphi(\lambda p, \lambda b) = \varphi(p, b)$ for all $p, b$ and $\lambda > 0$,

$(BI)$ $p \cdot x = b$ for all $x \in \varphi(p, b)$,

$(UHC)$ $\varphi$ is upper hemi-continuous at all $p, b \gg (0, 0)$,

$(Hv)$ $v(\lambda p, \lambda b) = v(p, b)$ for all $p, b$ and $\lambda > 0$,

$(Mv)$ $v$ is strictly increasing in $b$ and non-increasing in $p_h$ for all $h$,

$(Qv)$ The set $\{(p, b) : v(p, b) \leq \overline{v}\}$ is convex for all $\overline{v}$,

$(Cv)$ $v$ is continuous.
The dual problem to that of utility maximization is

\[
\min_x p \cdot x \\
\text{s.t. } x \in X \quad (EM) \\
u(x) \geq u
\]

**Definition:** The set of all solutions of (EM) for a given \((p, u)\) is denoted \(h(p, u)\), and \(h\) is called the Hicksian demand correspondence. For \(x \in h(p, u)\), \(e(p, u) = p \cdot x\) is called the expenditure function.
Proposition 2.3

If $\succeq$ is locally non-satiated and $u$ continuous on $X = \mathbb{R}_+^H$, then

1. \((Hh)\) $h(\lambda p, u) = h(p, u)$ for all $p, u$ and $\lambda > 0$,
2. \((UHCh)\) $h$ is upper hemi-continuous at all $(p, u) \gg (0, 0)$,
3. \((He)\) $e(\lambda p, u) = \lambda e(p, u)$ for all $p, u$ and $\lambda > 0$,
4. \((Me)\) $e$ is strictly increasing in $u$, and non-decreasing in $p_h$ for all $h$,
5. \((Conc e)\) $e$ is concave in $p$,
6. \((Ce)\) $e$ is continuous.
Proposition 2.4

(“Shephard’s Lemma”): If $\succeq$ is locally non-satiated and strictly convex, and $u$ continuous on $X = \mathbb{R}^H_+$, then $e$ is differentiable almost everywhere, and

$$\partial_p e(p, u) = h(p, u)^T$$

for almost all $(p, u) \gg (0, u(0))$.

Proof.

Fix $(p^0, u^0)$. For every $p$ define $Z(p) = p \cdot h(p^0, u^0) - e(p, u^0)$. Total expenditure under $(p, u^0)$ is minimized by $h(p, u^0)$, hence $Z(p) \geq 0$ for all $p$. Furthermore, $Z(p^0) = 0$.

Hence, if $Z$ is differentiable w.r.t. $p$ at $p^0$, then

$$\partial_p Z(p^0) = h(p^0, u^0)^T - \partial_p e(p^0, u^0) = 0.$$

Since $e$ is concave, it is differentiable at almost every $p^0$, and so is $Z$. \qed
Remarks:

- Price change has two impacts on expenditure: a direct effect of the order of magnitude $h(p, u)$, and an indirect effect through the induced change in demand. At the optimum of the (EM) problem, this indirect effect is of second order.

- Standard, less general, proof: assume that $e$ is continuously differentiable and use the envelope theorem: At $(p, u)$ with $h(p, u) \gg 0$

$$\partial_p e(p, u) = \partial_p (p \cdot h(p, u)) = h(p, u)^T + (p \cdot \partial_p h(p, u))^T.$$  

Since the utility constraint in the EM problem binds by the continuity of $u$ and by local non-satiation, one has by Lagrange $p - \lambda \partial_x u(x) = 0$ for a $\lambda \geq 0$. Hence, differentiating the identity $u(h(p, \bar{u})) = \bar{u}$ shows that $p \cdot \partial_p h(p, u) = 0$. 

Proposition 2.5

\( f \succeq \) is locally non-satiated and strictly convex, \( u \) continuous on \( X = \mathbb{R}^H_+ \), and \( h \) continuously differentiable, then for all
\( (p, u) \gg (0, u(0)) \)

(i) \( \partial_p h(p , u) \) is symmetric,
(ii) \( \partial_p h(p , u) \) is negative semi-definite,
(iii) \( \partial_p h(p , u)p = 0 \).

Proof.

Proposition (2.4) \( \Rightarrow \partial_p h(p , u) = \partial_{pp}^2 e(p , u) \). \( h \) is continuously differentiable \( \Rightarrow e \) is twice continuously differentiable. Hence (“Schwarz’s Theorem” in calculus), \( \partial_{pp}^2 e(p , u) \) is symmetric. Since \( e \) is concave in \( p \) (Proposition 2.3), \( \partial_{pp}^2 e(p , u) \) is negative-semidefinite. Differentiating the identity \( h(\lambda p , u) = h(p , u) \) with respect to \( \lambda \) and setting \( \lambda = 1 \) yields (iii).
Interpretation: \((\partial_p h(p, u))_{ij}\) : marginal change of consumption of good \(i\) with change of price of good \(j\) and constant utility \(u\).
\[\Rightarrow \partial_p h(p, u)\] similar to Slutsky matrix
2.5 The Relationship Between Utility Maximization and Expenditure Minimization

Proposition 2.6:

If $\succeq$ is locally non-satiated and $u$ continuous on $X = \mathbb{R}^H_+$, then for all $\bar{u} > u(0)$, $b > 0$, $p \in \mathbb{R}^H_+$,

(i) $\varphi(p, b) = h(p, v(p, b))$
(ii) $h(p, \bar{u}) = \varphi(p, e(p, \bar{u}))$
(iii) $e(p, v(p, b)) = b$
(iv) $v(p, e(p, \bar{u})) = \bar{u}$.

Proof.

For all $p \gg 0$, straightforward use of the definitions (including continuity of $u$ and local non-satiation). For $p$ on the boundary, continuity. (Exercise)
Proposition 2.7

If $\succeq$ is locally non-satiated and strictly convex, $u$ continuous on $\mathbb{R}_+^H$, and $\varphi$ is differentiable on $\mathbb{R}_{++}^H$, then $h$ is differentiable at almost every $(p, u) \succ (0, u(0))$ and

$$
\partial_p h(p, u(p, b)) = \partial_p \varphi(p, b) + \partial_b \varphi(p, b) \varphi(p, b)^T.
$$

Proof.

Differentiating (ii) in Proposition 2.6 with respect to $p$ yields, for $(p, u) \succ (0, u(0))$

$$
\partial_p h(p, u) = \partial_p \varphi(p, e(p, u)) + \partial_b \varphi(p, e(p, u)) \partial_p e(p, u)
$$

$$
= \partial_p \varphi(p, e(p, u)) + \partial_b \varphi(p, e(p, u)) h(p, u)^T,
$$

by Proposition 2.4. Since the budget $b = e(p, u)$ allows to achieve $u = \nu(p, b)$ by Proposition 2.6 (iii), the proof is complete.
Result: $\partial_p h$, at $u = v(p, b)$, is just the Slutsky matrix of Proposition 1.3. This is somewhat intuitive from what has been said before, but not trivial.
Proposition 2.5 ⇒ Slutsky matrix symmetric
⇒ A Marshallian demand system with asymmetric Slutsky matrix cannot be derived from preference-based optimization.
2.6 Integrability

- utility maximization $\rightarrow$ Marshallian demand
- **Question**: Marshallian demand $\rightarrow$ utility maximization
- **Immediate**: Marshallian demand has to satisfy Propositions 2.2-2.7
- **Goal**: recover utility function from Marshallian demand
- Best way to proceed: through the expenditure function
From Propositions 2.4 and 2.6:

\[ \partial_p e(p, u^0)^T = \varphi(p, e(p, u^0)) \]  \hfill (3)

This is a system of differential equations in \( p \) for the unknown \( e \) with parameter \( u^0 \).

Choose \( p^0, b^0 \) rather than \( u^0 \), take \( x^0 = \varphi(p^0, b^0) \) and \( u^0 = u(x^0) \).

Use

\[ e(p^0, u^0) = p^0 \cdot \varphi(p^0, b^0) \]  \hfill (4)

as an initial condition for (3).

The theory of partial differential equations now implies that (3) has a solution, locally around \( p^0 \), if and only if the Slutsky matrix, \( \partial_p \varphi + \partial_b \varphi \varphi^T \), is symmetric

\( e \) concave iff Slutsky matrix negative semi-definite \( \Leftrightarrow \) (WARP)
(BI), (H), (WARP), and symmetry of the Slutsky matrix of $\varphi$ are equivalent to preference optimization.
Recovering preferences from observed data is useful for empirical work. Examples

- When is two-stage budgeting (Chapter 1.5) valid?

  Demand system $\varphi$ separable if utility function separable
  \[ (u(x_1, \ldots, x_H) = F(u^1(q_1), \ldots, u^L(q_L)), \text{where the } q_i \text{ are commodity vectors}) \]

  In order for each commodity group to have one single group price index (determination of group budgets $b^i$ in step 1 of the process), it is sufficient (and almost necessary) that intra-group preferences are homothetic.

- Preferences of a consumer with linear Engel curves (important for aggregation)? $\rightarrow$ "Gorman polar form":

  \[ e(p, u) = \alpha(p) + u\beta(p) \]

  where $\alpha$ and $\beta$ concave and homogenous of degree 1.
2.7 Welfare

Does a given price/budget manipulation make the (hopefully representative) household “better” or “worse” off? see Mas-Colell, Whinston, Green, Ch. 3.I and the exercises